Exponential Stability of the PDAF with a Modified Riccati Equation in a Cluttered Environment

Yong-Shik Kim and Keum-Shik Hong

Abstract: The probabilistic data association filter (PDAF) is known to provide better tracking performance than the standard Kalman filter (KF) in a cluttered environment. In this paper, the stability of the PDAF of Fortmann *et al.* [7], in the presence of uncertainties with regard to the origin of measurement, is investigated. The modified Riccati equation derived by approximating two random terms with their expectations is used to prove the stability of the PDAF. A new Lyapunov function based approach, which is different from the quantitative evaluation of Li and Bar-Shalom [17], is pursued. With the assumption that the system and observation noises are bounded, specific tracking error bounds are established.

Keywords: Target tracking, probabilistic data association filter, stability, state estimation, Kalman filter, Lyapunov function

I. Introduction

The target tracking problem refers to the process of estimating the state of a target using a set of measurements associated with the target. In a cluttered environment, a measurement may have been originated from any one of the following: the target of interest, interfering objects, clutter, countermeasures, or false alarms in the detection process. The uncertainty with regard to the origin of a measurement makes the tracking problem much more difficult than a regular estimation problem. Accordingly, how to overcome, or manifest, the vagueness of the origins of data, which is referred to as a data association problem, is the crux of a tracking problem. A typical algorithm in this category is the probabilistic data association filter (PDAF) [1][2][5][7][14][17]. The first work on the PDAF, which is more complex than the standard KF [8][13], was originally introduced by Bar-Shalom and Tse in 1975 [1].

While the PDAF has demonstrated a good tracking performance in the presence of uncertainties, neither the stability proof nor the convergence analysis of the PDAF has been completed yet. On the other hand, the stability and the convergence analyses of the KF algorithms for nonlinear and/or time-varying systems are still widely investigated in the literature [4][10]-[12][18][19][21]. For linear systems, the standard Riccati equation leads to the stability of the KF, provided appropriate controllability and observability conditions hold [8] [13] and the KF's prediction covariance always converges in the steady state.

A PDAF algorithm was introduced by Fortmann *et al.* [7], in which a deterministic Riccati equation, as an approximate propagation of the average covariance matrix, was derived by replacing the random terms in the original equation with their expectations over all possible validated measurements. The stability of the tracking algorithm in [7] depends critically on the detection and false alarm probabilities. It is apparent from their work that the modified Riccati equation, which consists of target detection probability and false alarm probability, converges to the steady-state covariance in most cases. However, the existence of a region in which the equation diverges is also apparent. Therefore, the stability issue of the PDAF was not answered completely. Moreover, the approach in [7] may not be suitable for tracking in a heavily cluttered environment.

Li and Bar-Shalom [17] introduced another PDAF. The approach of [17] is hybrid in the sense that a continuous-valued covariance matrix, as a function of a discrete-valued random variable, is used to characterize the performance of the algorithm considered. The covariance matrix is calculated off-line recursively using the modified Riccati equation, which is derived by replacing the measurement-dependent terms of the original stochastic Riccati equation with their conditional expectations evaluated only over possible locations of measurements in the validation region. The dependence of the covariance matrix on the number of validated measurements, a discrete-valued random variable, is retained after the expectation operation. The approach of [17] has the merit that it yields a quantification of the transients of tracking divergence as well as substantially better accuracy than the approach of [7]. However, an analytic proof of the stability and the boundedness of the tracking error in a heavily cluttered environment are not yet provided in [17]. This paper is basically motivated by the lack of a stability proof of the PDAF.

The exponential stability of the linear Kalman filter for estimating time-varying parameters of a linear regression model, in which the regressors are stochastic and nonstationary, was investigated in [10]. The conditions and techniques used in [10] are different from the traditional ones in the areas of system identification and adaptive signal processing. In this paper, the approach of [10] is utilized in proving the stability of the PDAF with a modified Riccati equation.

The main contributions of this paper are: The stability of the PDAF algorithm with a modified Riccati equation for estimating the state of stochastic dynamic model, in the presence of uncertainties of the measurement origin, is investigated. It is shown that if the observation sequence belongs to a gate s-algebra (defined in Section III), the information reduction factor is chosen adequately between 0 and 1, and the system and observation noises are bounded, then the stability of the modified PDAF is guaranteed. A new approach based on a Lyapunov function, which is different from the quantitative evaluation in [17], is proposed. Finally, specific tracking error

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bounds for given bounds of the system and observation noises are established.

This paper is organized as follows:

In Section II, the standard Kalman filter algorithm and the modified PDAF algorithm are compared. The problem formulation is provided in Section III. The main stability analysis is carried out in Section IV. In Section V, conclusions are stated.

II. KF vs PDAF with a modified Riccati equation

To enhance understanding of the issues of this paper, the standard KF and the PDAF with a modified Riccati equation are compared in this section. Consider the following state-space representation of the target motion and observation:

$$x_{k+1} = F_k x_k + w_k, \quad k \ge 0, \tag{2.1}$$

$$y_k = H_k x_k + n_k, \quad k \ge 1, \tag{2.2}$$

where $x_k \in R^d$ and $y_k \in R^c$ are the state and observation vectors, respectively, w_k and n_k are mutually uncorrelated white Gaussian noise vectors with zero mean and covariances Q and R, respectively, and F_k and H_k are assumed to be known time-varying system and observation matrices. Initial state x_0 is assumed to be Gaussian and uncorrelated with the system and observation noises w_k and n_k. It is assumed that system (2-1) and (2-2) is uniformly completely observable, see [13, p.232] or [4][18][19] for the definition of the uniform complete observability.

The two algorithms are now summarized as follows:

1. KF algorithm [13, p.200]

The state estimate equation for (2.1)-(2.2) is

$$\hat{x}_{k|k-1} = F_{k-1}\hat{x}_{k-1|k-1}, \tag{2.3}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - H_k\hat{x}_{k|k-1}), \tag{2.4}$$

where $\hat{x}_{k|k-1}$ is the state estimate at time k conditioned on measurement data up to time $k-1$; $K_k \overset{\Delta}{=} P_{k|k-1}H_k'[R + H_k P_{k|k-1} H_k']^{-1}$ is the KF gain matrix at time k. The associated covariance equation is

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}' + Q, \tag{2.5}$$

$$P_{k|k} = P_{k|k-1} - K_k S_k K'_k. \tag{2.6}$$

where $P_{k|k-1}$ is the covariance matrix of the state error $\tilde{x}_{k|k-1} \overset{\Delta}{=} x_k - \hat{x}_{k|k-1}$; S_k is the covariance matrix of the innovation term $h_k \overset{\Delta}{=} y_k - H_k\hat{x}_{k|k-1}$.

2. PDAF Falgorithm with a modified Riccati equation [2, p.213]

The state estimate equation is

$$\hat{x}_{k|k-1} = F_{k-1}\hat{x}_{k-1|k-1}, \tag{2.7}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k b_{k,i}(y_{k,i} - H_k\hat{x}_{k|k-1}), \tag{2.8}$$

where K_k takes the same form as the KF gain matrix in (2.4); $b_{k,i}$ is the *a posterior* probability for $y_{k,i}$ to be target-originated, where $y_{k,i}$ is the i-th validated measurement at time k. The modified Riccati equation, which corresponds to the covariance equation of the KF, is

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}' + Q, \tag{2.9}$$

$$P_{k|k} = P_{k|k-1} - q_2 K_k S_k K_k', \tag{2.10}$$

where S_k in (2.10) is the covariance matrix of the innovation term $h_k \overset{D}{=} \sum_{i=1}^{z_k} b_{k,i}(y_{k,i} - H_k\hat{x}_{k|k-1})$; z_k is the number of validated measurements at time k; q_2 is the information reduction factor to be defined in Section III next, see [2, 7].

Remark 1: (2.9)-(2.10) are deterministically approximated prediction and update equations of the covariance matrix, which utilize the averaged covariance matrix obtained by replacing random terms of the PDAF with their expectations over all possible validated measurements.

III. Problem formulation

In many tracking problems, uncertainties in the target motion and in the measured values are usually modeled as additive random noises. The covariance matrices of the process and measurement noises specify the uncertainties in target motion and measured values, respectively. In practice, when tracking a target in clutter, however, more than one measurement are possibly available at any time step, and therefore the optimal estimate does not hold anymore unless a correct and complete target-observation assignment is accomplished at each time step. In this situation, the tracking performance depends not only upon the noise covariance but also upon the amount of uncertainty in measurement origin. This dependence is characterized in terms of the probabilities of detection and false alarm. The dependence of error covariance upon the detection and false alarm probabilities is explicitly characterized by a scalar parameter q_2 in the modified Riccati equation.

Consider the PDAF with a modified Riccati equation (2.7)-(2.10) for estimating the state of (2.1)-(2.2) in the following form:

$$\begin{aligned}\hat{x}_{k+1} &= F_k\hat{x}_k + F_k P_k H_{k+1}'[H_{k+1}P_k H_{k+1}' + R]^{-1}\sum_{i=1}^{z_{k+1}} b_{k+1,i}(y_{k+1,i} - H_{k+1}F_k\hat{x}_k) \\ &= F_k\hat{x}_k + F_k P_k H_{k+1}'[H_{k+1}P_k H_{k+1}' + R]^{-1}(y_{k+1} - H_{k+1}F_k\hat{x}_k),\end{aligned} \tag{3.1}$$

$$P_{k+1} = F_k P_k F_k' - q_2 F_k P_k H_{k+1}'[R + H_{k+1}P_k H_{k+1}']^{-1}H_{k+1}P_k F_k' + Q, \tag{3.2}$$

where $y_{k+1} = \sum_{i=1}^{z_{k+1}} b_{k+1,i}y_{k+1,i}$, P_0 is a symmetric positive definite matrix, and R and Q are positive definite matrices. R and Q may be regarded as *a priori* estimates for the variances of n_k and w_k, respectively.

The following notation and terminology are introduced: For a matrix X, $l_{\max}(X)$ and $l_{\min}(X)$ denote the maximum and minimum eigenvalues of X and the induced norm is $\| X \| \overset{D}{=} \sqrt{l_{\max}(XX')}$, where $'$ denotes the transposition.

Regarding the PDAF algorithm with a modified Riccati equation, let Y_k denote the cumulative observation sequence set consisting of all measurements up to time k such that

$Y^k = \{\sum_{i=1}^{z_k} y_{j,i}\}_{j=1}^{k}$. First, assume that the true measurement at time $k+1$, conditioned upon Y^k, is normally distributed, i.e.,

$$p[y_{k+1} | Y^k] = N[y_{k+1}; H_{k+1}F_k \hat{x}_k, S_{k+1}].$$

A region in the measurement space, where the measurement will have some (high) probability, is defined as follows:

$$\begin{aligned}\tilde{\Lambda}_{k+1}(c) &= \{y_{k+1}: [y_{k+1} - H_{k+1}F_k\hat{x}_k]'S_{k+1}^{-1}[y_{k+1} - H_{k+1}F_k\hat{x}_k] \le c\} \\ &= \{y_{k+1}: h'_{k+1}S_{k+1}^{-1}h_{k+1} \le c\},\end{aligned}$$

where c is a threshold parameter to be selected beforehand and h is the innovation term [2]. The region defined above is called the validation region or the gate. It is an ellipse of minimum volume. Related to a validation region, the following definition is introduced.

Definition 1: A s -algebra on an abstract set Θ is a collection of subsets of Θ which contains the null set f and is closed under countable set operations. If the set Θ assumes values in a certain validation region or a gate, it is particularly called a gate s -algebra.

In the PDAF, it is assumed that the correct measurement is detected with probability P_D and that all other measurements are Poisson-distributed with parameter $C_f V_g$, where V_g is the volume of the validation gate and C_f is the expected number of false measurements per unit volume [2, 7, 17]. Also, the information reduction factor q_2 depends upon the probabilities of detection and false alarm, and also upon the volume of the data association gate as follows:

$$q_2 = q_2(P_D, C_f V_g).$$

The following assumptions are now made.

A1: Assume that there exists a constant $d > 0$ and an integer $h > 0$ such that

$$E\left\{\sum_{k=mh}^{(m+1)h-1} \frac{F_k' H_{k+1}' H_{k+1} F_k}{1 + \|H_{k+1}\|^2} \,\Big|\, \Im_{mh-1}\right\} \ge d\,I \quad \text{almost surely}, \quad \forall m \ge 0$$

which $\Im_{mh-1}$ is the gate s -algebra generated by $\{y_0, \cdots, y_{mh-1}\}$.

A2: $\{n_k, w_k\}$ is random or deterministic process satisfying

$$s_r = \sup_k E\{\|n_k\|^r + \|w_k\|^r \mid \Im_{k-1}\} < \infty, \text{ for some } r > 4,$$

$$m_4 = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\{\|n_k\|^4 + \|w_k\|^4\} < \infty, \text{ almost surely.}$$

A3: The observation sequence y_k belongs to a gate s -algebra and the information reduction factor q_2 is chosen adequately assumes a value between 0 and 1.

Remark 2: Assumption A1 assures the observability of the system considered. Assumption A2 characterizes the system and observation noises. Assumptions A1 and A2 are adopted from [10], see also [22] or [9, p. 372-374]. Note that assumption A1 is weaker than the uniform complete observability condition [20][22] because an upper bound is not required. Finally, assumption A3 characterizes the uncertainty of measurement origin.

Remark 3: It is known that if the noises $\{w_k, n_k\}$ is white Gaussian and assumption A3 holds, then $\hat{x}_k$ generated by (3.1) and (3.2) is the best estimate for x_k with estimation error covariance P_k [2][10], i.e., let $\tilde{x}_k \overset{\Delta}{=} x_k - \hat{x}_k$, then

$$\hat{x}_k = E[x_k | \Im_{k-1}], \; P_k = E[\tilde{x}_k \tilde{x}_k' | \Im_{k-1}], \tag{3.3}$$

provided $E[w_k | \Im_{k-1}] = E[n_k | \Im_{k-1}] = 0$, $Q = E[w_k w_k' | \Im_{k-1}]$, $R = E[n_k n_k' | \Im_{k-1}]$, $\hat{x}_0 = E[x_0]$ and $P_0 = E[\tilde{x}_0 \tilde{x}_0']$, in which $\Im_{k-1}$ is the gate s -algebra generated by $\{y_0, \cdots, y_{k-1}\}$.

The main theorem of this paper is now stated. In Section IV, it will be shown that the above conditions are the best possible.

Theorem 1: Let assumptions A1-A3 hold. Then, for $\{\hat{x}_k\}$ given by (3.1)-(3.2),

1) $\limsup_{n\to\infty} E\|\hat{x}_n - x_n\|^2 \le C_1 q_2 [s_r]^{2/r}$, and

2) $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\|\hat{x}_i - x_i\| \le C_2 (q_2)^{1/2} [m_4]^{1/4}$,

almost surely, where s_r, m_4 and r are defined in assumption A2, and q_2 is an information reduction factor introduced in A3. C_1 and C_2 are deterministic constants.

Remark 4: Theorem 1 asserts that the stability of the PDAF with a modified Riccati equation is guaranteed if the system is uniformly completely observable, the process and observation noises are bounded, and all the measurements in a cluttered environment belong to a validation region established in the detector. It is known that the PDAF, in a cluttered environment, shows better tracking performance than the standard Kalman filter. This is because the uncertainty of the measurement origin can be adjusted by introducing q_2.

IV. Main results: Stability proof

Before proving Theorem 1 above, the following lemmas are stated:

Lemma 1: Let $Q_k = F_k P_k F_k' - q_2 F_k P_k H_{k+1}'(H_{k+1} P_k H_{k+1}' + R)^{-1} H_{k+1} P_k F_k'$. Let $\Psi_{k,i}$ be the transition matrix associated with $\{F_k\}_{j=i}^{k-1}$ [15, 16] such that

$$\Psi_{k,i} = F_{k-1}\Psi_{k-1,i} = \cdots = F_{k-1}\cdots F_i \Psi_{i,i}, \quad \Psi_{i,i} = I, \quad \forall k-1 \ge i \ge 0.$$

Let assumption A3 hold. Then, for any integers $m \ge 0$, $h > 0$, and $k \in [mh, (m+1)h]$, where h is an arbitrary but fixed integer which breaks the time axis into blocks of length h, the following inequality holds:

$$\begin{aligned} tr(Q_k^4) \le\; & tr(\Pi_{mh}^4) - tr\{q_2 \Gamma_{mh}^5 F_k' H_{k+1}' (R + H_{k+1}\Gamma_{mh}H_{k+1}')^{-1} H_{k+1} F_k\} \\ & + O(tr(\Pi_{mh}^3)) + O(1) \end{aligned}$$

where $\Pi_{mh} \overset{\Delta}{=} \Psi_{(m+1)h+1,mh} P_{mh} \Psi'_{(m+1)h+1,mh}$ and

$$\Gamma_{mh} \overset{\Delta}{=} \Pi_{mh} + \sum_{k=mh+1}^{k=mh+h+1} \Psi_{k-1,mh} Q \Psi'_{k-1,mh}.$$

Proof: First of all, note that Q_k is positive semi-definite for all $k \ge 0$, i.e.,

$$Q_k = F_k P_k F_k' - q_2 F_k P_k H_{k+1}'(R + H_{k+1} P_k H_{k+1}')^{-1} H_{k+1} P_k F_k'$$

$$=[(F_kP_kF_k')^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)H_{k+1}P_kH_{k+1}' + q_2^{-1}R\}^{-1}(H_{k+1}F_k^{-1})]^{-1}\geq 0 \tag{4.1}$$

where the matrix inversion formula has been used to drive the second equality. The covariance equation of the PDAF with a modified Riccati equation of (3.2) at time k is

$$\begin{aligned}P_k &= F_{k-1}P_{k-1}F_{k-1}' - q_2F_{k-1}P_{k-1}H_k'(R+H_kP_{k-1}H_k')^{-1}H_kP_{k-1}F_{k-1}'+Q\\ &\leq F_{k-1}P_{k-1}F_{k-1}'+Q\\ &\leq F_{k-1}F_{k-2}P_{k-2}F_{k-2}'F_{k-1}'+F_{k-1}QF_{k-1}'+Q\\ &\vdots\\ &\leq F_{k-1}F_{k-2}\cdots F_1F_0P_0F_0'F_1'\cdots F_{k-2}'F_{k-1}'+Q+F_{k-1}QF_{k-1}'+\cdots\\ &\quad + F_{k-1}F_{k-2}\cdots F_1F_0QF_0'F_1'\cdots F_{k-2}F_{k-1}'.\end{aligned}$$

Now, to simplify the above expression, the state transition matrix $\emptyset_{k,i}$ defined in the statement of Lemma 1 is utilized as follows:

$$\begin{aligned}P_k &\leq \Psi_{k,0}P_0\Psi_{k,0}'+\Psi_{k,k}Q\Psi_{k,k}'+\Psi_{k,k-1}Q\Psi_{k,k-1}'+\cdots+\Psi_{k,0}Q\Psi_{k,0}'\\ &\overset{\Delta}{=}\Psi_{k,0}P_0\Psi_{k,0}'+\textstyle\sum_{i=0}^{k}\Psi_{k,i}Q\Psi_{k,i}'.\end{aligned}$$

And therefore

$$F_kP_kF_k' \leq \Psi_{k+1,0}P_0\Psi_{k+1,0}'+\textstyle\sum_{i=0}^{k+1}\Psi_{k+1,i}Q\Psi_{k+1,i}'.$$

In particular, for any $k\in[mh,(m+1)h]$

$$\begin{aligned}F_kP_kF_k' &\leq F_{mh+h}F_{mh+h-1}\cdots F_{mh+1}F_{mh}P_{mh}F_{mh}'F_{mh+1}'\cdots F_{mh+h-1}'F_{mh+h}'\\ &\quad+\textstyle\sum_{k=mh+1}^{k=mh+h+1}\Psi_{k-1,mh}Q\Psi_{k-1,mh}'\\ &\overset{\Delta}{=}\Pi_{mh}+\textstyle\sum_{k=mh+1}^{k=mh+h+1}\Psi_{k-1,mh}Q\Psi_{k-1,mh}'\\ &\overset{\Delta}{=}\Gamma_{mh}.\end{aligned} \tag{4.2}$$

Now, for matrices L and X such that $0\leq L\leq X$, the following inequalities holds:

$$tr[L^4]\leq tr[LX^3]\leq tr[X^4]\ .$$

Hence, by (4.2), (3.1) and (3.2) we have

$$\begin{aligned}tr[Q_k^4] &= tr[\{(F_kP_kF_k')^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)H_{k+1}P_kH_{k+1}'\\ &\quad+q_2^{-1}R\}^{-1}(H_{k+1}F_k^{-1})\}^{-1}]^4\\ &\leq tr[\{\Gamma_{mh}^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)H_{k+1}P_kH_{k+1}'+q_2^{-1}R\}^{-1}\\ &\quad\cdot(H_{k+1}F_k^{-1})\}^{-1}]^4\\ &= tr[\{\Gamma_{mh}^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)H_{k+1}P_kH_{k+1}'+q_2^{-1}R\}^{-1}\\ &\quad\cdot(H_{k+1}F_k^{-1})\}^{-1}\{\{\Gamma_{mh}^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)\\ &\quad\cdot H_{k+1}P_kH_{k+1}'+q_2^{-1}R\}^{-1}(H_{k+1}F_k^{-1})\}^{-1}\}^3]\\ &= tr[\{\Gamma_{mh}^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)H_{k+1}P_kH_{k+1}'+q_2^{-1}R\}^{-1}\\ &\quad\cdot(H_{k+1}F_k^{-1})\}^{-1}\{\Gamma_{mh}-q_2F_k\Gamma_{mh}H_{k+1}'\\ &\quad\cdot(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\Gamma_{mh}F_k'\}^3]\\ &\leq tr[\{\Gamma_{mh}^{-1}+(H_{k+1}F_k^{-1})'\{(q_2^{-1}-1)H_{k+1}P_kH_{k+1}'+q_2^{-1}R\}^{-1}\\ &\quad\cdot(H_{k+1}F_k^{-1})\}^{-1}\Gamma_{mh}^3]\\ &= tr[\Gamma_{mh}^3\{\Gamma_{mh}-q_2F_k\Gamma_{mh}H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}\\ &\quad\cdot H_{k+1}\Gamma_{mh}F_k'\}]\\ &\leq tr(\Gamma_{mh}^4)-tr\{q_2\Gamma_{mh}^3F_k\Gamma_{mh}H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}\\ &\quad\cdot H_{k+1}\Gamma_{mh}F_k'\}.\end{aligned} \tag{4.3}$$

Now, the following inequality is claimed:

$$\begin{aligned}&tr\{q_2\Gamma_{mh}^3F_k\Gamma_{mh}H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\Gamma_{mh}F_k'\}\\ &\quad-tr\{q_2\Gamma_{mh}^5F_k'H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}F_k\}\geq 0.\end{aligned} \tag{4.4}$$

(4.4) can be easily shown by using Lemma 1.7 and Lemma 1.10 of [6]. The first term of (4.4) can be rewritten as follows:

$$\begin{aligned}&q_2tr\{\Gamma_{mh}^{3/2}F_k\Gamma_{mh}H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\Gamma_{mh}F_k'\Gamma_{mh}^{3/2}\}\\ &\leq q_2tr(\Gamma_{mh}^{3/2}\Gamma_{mh}^{3/2})\ tr(F_k\Gamma_{mh}\Gamma_{mh}F_k')\ tr\{H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\}\\ &\leq q_2tr(\Gamma_{mh}^3)\ tr(F_k\Gamma_{mh}^2F_k')\ tr\{H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\}\\ &\leq q_2tr(\Gamma_{mh}^3)\ tr(\Gamma_{mh}^2)\ tr(F_kF_k')\ tr\{H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\}.\end{aligned}$$

Similarly, the second term of (4.4) can be rewritten as follows:

$$\begin{aligned}&q_2tr\{\Gamma_{mh}^{5/2}F_k'H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}F_k\Gamma_{mh}^{5/2}\}\\ &\leq q_2tr(\Gamma_{mh}^{5/2}\Gamma_{mh}^{5/2})tr(F_kF_k')tr\{H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\\ &\leq q_2tr(\Gamma_{mh}^5)\ tr(F_kF_k')\ tr\{H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}\}.\end{aligned}$$

Since the Holder inequality such that

$$tr\{X^3\}tr\{X^2\}\leq b^4tr\{X^5\},\quad b\geq 1$$

holds for any b-dimensional nonnegative definite matrix X, $tr(\Gamma_{mh}^3)tr(\Gamma_{mh}^2)\leq tr(\Gamma_{mh}^5)$ by taking $b=1$, Hence (4.4) is obtained. Therefore, by substituting (4.4) into (4.3), the following inequality is derived.

$$tr(Q_k^4)\leq tr(\Gamma_{mh}^4)=tr\{q_2\Gamma_{mh}^5F_k'H_{k+1}'(H_{k+1}\Gamma_{mh}H_{k+1}'+R)^{-1}H_{k+1}F_k\} \tag{4.5}$$

Now, by applying the Holder's inequality, the first term in the right hand side of (4.5) can be written as [10]

$$tr(\Gamma_{mh}^4)=tr(\Pi_{mh}^4)+O(tr(\Pi_{mh}^3))+O(1). \tag{4.6}$$

Consequently, by substituting (4.6) into (4.5), the assertion of Lemma 1 is obtained. ■

The result of Lemma 1 will be used in proving Lemma 2 next. To guarantee the stability of a stochastic system, the stability of moments as well as a sample should be assured. Now let's prove the boundedness of fourth-order moment of the state error covariance.

Lemma 2: Under assumptions A1 and A3,

$$\sup_k E\|P_k\|^4<\infty.$$

Proof: First, define that $T_m\overset{\Delta}{=}\sum_{k=(m-1)h}^{mh-1}tr(P_{k+1}^4)$, for $m\geq 1$. Then, by (3.2), Lemma 1, and Holder's inequality we have

$$\begin{aligned}T_{m+1} &= \sum_{k=mh}^{(m+1)h-1}tr(P_{k+1}^4)=\sum_{k=mh}^{(m+1)h-1}tr(Q_k+Q)^4\\ &\leq \sum_{k=mh}^{(m+1)h-1}\{tr(Q_k^4)+O(tr(Q_k^3))+O(1)\}\end{aligned}$$

$$= h\,tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - \sum_{k=mh}^{(m+1)h-1} tr\{q_2\,\Gamma_{mh}^5 F_k' H_{k+1}' (R + H_{k+1}\Gamma_{mh} H_{k+1}')^{-1} \cdot H_{k+1} F_k\}$$

$$= h\,tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - q_2 tr\{\Gamma_{mh}^5 \cdot \sum_{k=mh}^{(m+1)h-1} F_k' H_{k+1}' (R + H_{k+1}\Gamma_{mh} H_{k+1}')^{-1} \cdot H_{k+1} F_k\}$$

$$\le h\,tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - q_2 tr\left\{\Gamma_{mh}^5 \cdot \sum_{k=mh}^{(m+1)h-1} \frac{F_k' H_{k+1}' H_{k+1} F_k}{\|\{R + l_{\max}(\Gamma_{mh})\}(1 + \|H_{k+1}\|^2)\|}\right\}$$

$$\le h\,tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - \frac{q_2}{\|R + l_{\max}(\Gamma_{mh})\|} tr\left\{\Gamma_{mh}^5 \cdot \sum_{k=mh}^{(m+1)h-1} \frac{F_k' H_{k+1}' H_{k+1} F_k}{(1 + \|H_{k+1}\|^2)}\right\}. \quad (4.7)$$

Thus, by taking conditional expectations and using Holder inequality,

$$E[T_{m+1} | \Im_{mh-1}] \le h\;\; tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - \frac{q_2}{\|R + l_{\max}(\Gamma_{mh})\|} \cdot tr\left\{\Gamma_{mh}^5 \cdot E\left(\sum_{k=mh}^{(m+1)h-1} \frac{F_k' H_{k+1}' H_{k+1} F_k}{1 + \|H_{k+1}\|^2} \,|\, \Im_{mh-1}\right)\right\}$$

$$\le h\,tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - \frac{q_2 d}{\|R + l_{\max}(\Gamma_{mh})\|} tr(\Gamma_{mh}^5)$$

$$\le h tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - \frac{q_2 d\; tr(\Gamma_{mh})}{d^4 \|R + l_{\max}(\Gamma_{mh})\|} tr(\Gamma_{mh}^4)$$

$$\le h tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1) - \frac{q_2 d\; h \|Q\|}{d^4 (\|R\| + h\|Q\|)} tr(\Pi_{mh}^4)$$

$$= \left(1 - \frac{q_2 d\; \|Q\|}{d^4 (\|R\| + h\|Q\|)}\right) h\,tr(\Pi_{mh}^4) + O(tr(\Pi_{mh}^3)) + O(1). \quad (4.8)$$

In addition, it is evident from (3.2) and Holder inequality that

$$h\,tr(\Pi_{mh}^4) = \sum_{k=(m-1)h}^{mh-1} tr(\Pi_{mh}^4)$$

$$\le \sum_{k=(m-1)h}^{mh-1} tr\{(P_{k+1} + \Psi_{(2mh-h-1-k,mh-1)} Q \Psi_{(2mh-h-1-k,mh-1)}')^4\}$$

$$\le T_m + O\left(\sum_{k=(m-1)h}^{mh-1} tr(P_{k+1}^3)\right) + O(1)$$

$$\le T_m + O((T_m)^{3/4}) + O(1)$$

since $(T_m)^{3/4} = \left\{\sum_{k=(m-1)h}^{mh-1} tr(P_{k+1}^4)\right\}^{3/4}$. Substituting this into (4.8), it follows that

$$E[T_{m+1} | \Im_{mh-1}] \le \left(1 - \frac{q_2 d\; \|Q\|}{d^4 (\|R\| + h\|Q\|)}\right) T_m + O((T_m)^{3/4}) + O(1).$$

And applying the following elementary inequality

$$x^{3/4} \le ex + \left(\frac{3}{4e}\right)^3, \quad \forall x \ge 0, \quad \forall e > 0$$

for small e to the above equation, we get

$$E[T_{m+1} | \Im_{mh-1}] \le \left(1 - \frac{q_2 d\, \|Q\|}{2d^4 (\|R\| + h\|Q\|)}\right) T_m + O(1). \quad (4.9)$$

Consequently, by the smoothing property of expectation,

$$E\{E[T_{m+1} | \Im_{mh-1}]\} = E[T_{m+1}] \le \left(1 - \frac{d q_2 \|Q\|}{2d^4 (\|R\| + h\|Q\|)}\right) E[T_m] + O(1). \quad (4.10)$$

From this, it is not difficult to obtain the following result:

$$\sup_m E[T_m] < \infty.$$

■

Now, from the boundedness of fourth-order moment, we can prove that second-order moment of the state error covariance is also bounded.

Lemma 3: Under assumptions A2 and A3,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \|P_k\|^2 \le \infty, \qquad \text{almost surely.}$$

Proof: It follows, using the similar argument as in Lemma 1, that for any $k \in [mh, (m+1)h]$, $m \ge 0$

$$tr(Q_k^2) \le tr(\Pi_{mh}^2) + O(tr(\Pi_{mh})) + O(1) - tr\{q_2\,\Gamma_{mh}^3 F_k' H_{k+1}' (R + H_{k+1}\Gamma_{mh} H_{k+1}')^{-1} H_{k+1} F_k\}.$$

Consequently, we have the result similar to the proof of (4.10) in Lemma 2 as follows:

$$E[M_{m+1} | \Im_{mh-1}] \le \left(1 - \frac{d q_2 \|Q\|}{2d^2 (\|R\| + h\|Q\|)}\right) M_m + O(1), \quad \forall m \ge 0, \quad (4.11)$$

where

$$M_m = \sum_{k=(m-1)h}^{mh-1} tr(P_{k+1}^2).$$

Let us denote

$$g_{m+1} = M_{m+1} - E[M_{m+1} | \Im_{mh-1}], \qquad m \ge 0 \quad (4.12)$$

then, $\{g_m, \Im_{mh-1}, m \ge 0\}$ is martingale difference sequence, and satisfies

$$\sup_m E[g_m]^2 < \infty$$

by Lemma 2. Now by (4.11) and (4.12) it follows that

$$M_{m+1} = E[M_{m+1}|\Im_{mh-1}] + g_{m+1}$$
$$\le \left(1 - \frac{d q_2 \| Q\|}{2d^2[\| R\| + h\| Q\|]}\right) M_m + O(1) + g_{m+1.}$$

Summing up from 0 to $n-1$,

$$M_m \le M_0 - \frac{d q_2 \| Q\|}{2d^2[\| R\| + h\| Q\|]} \sum_{m=0}^{n-1} M_m + O(n) + \sum_{m=0}^{n-1} g_{m+1},$$

and so

$$\frac{1}{n}\sum_{m=0}^{n-1} M_m \le \frac{2d^2(\| R\| + h\| Q\|)}{q_2 d \| Q\|} \left\{ \frac{M_0}{n} + \frac{1}{n}\sum_{m=0}^{n-1} g_{m+1} + O(1) - \frac{M_n}{n} \right\}.$$

Thus, by the martingale convergence theorem and the Kronecker Lemma [3]

$$\limsup_{n\to\infty} \frac{1}{\mathrm{n}} \sum_{m=0}^{n-1} M_m < \infty, \qquad \text{almostsurely}$$

From this, it is easy to obtain the result of Lemma 3. ■

Third, we will deal with the exponential boundedness which is inevitable to Theorem 1.

Lemma 4 [10]: Let $\{a_k\}$ be an adapted sequence of the gate s -algebra $\Im_k$, $a_k \ge 1, \ \forall k \ge 0$. If for some integer $h > 0$, and constants $0 < a < 1, \ l < \infty$,

$$E[a_k \mid \Im_{k-1}] \le a a_{k-1} + l, \quad \forall k \ge 1, \quad 0 < a < 1, \quad 0 < l < \infty,$$

then there exist constants $g \in (0,1)$ and $M < \infty$ such that

$$E\left[\prod_{k=m}^{n}\left(1 - \frac{1}{a_k}\right)\right] \le M g^{\,n-m+1}, \quad \forall n \ge m, \ \forall m \ge 0.$$

Proof: See [10].

Now, we prove the most critical lemma in this paper.

Lemma 5: Let us now denote $\tilde{x}_k = x_k - \hat{x}_k$, and consider the following stochastic Lyapunov function V_k:

$$V_k = \tilde{x}_k' P_k^{-1} \tilde{x}_k, \tag{4.13}$$

then for any $k \ge 0$ and by A3,

$$V_{k+1} \le q_2^{-1} V_k - \frac{q_2^{-1} V_k}{4 + etr(q_2 F_k P_k F_k')}$$
$$+ O(\ \| q_2 F_k P_k F_k' \| \{ \| n_{k+1} \|^2 + \| w_k \|^2 \}\ )$$

where $e = 2\| Q^{-1}\|$.

Proof: Subtracting the second equation of (3.1) from (2.1) yields the error equation in the following form:

$$\tilde{x}_{k+1} = J_k \tilde{x}_k + z_{k+1} \tag{4.14}$$

where

$$J_k = F_k - K_{k+1} H_{k+1} F_k,$$
$$K_{k+1} = F_k P_k H_{k+1}' (R + H_{k+1} P_k H_{k+1}')^{-1},$$
$$z_{k+1} = -K_{k+1} n_{k+1} + w_k.$$

And rewrite (3.2) using K_{k+1} defined above as

$$P_{k+1} = (1 - q_2)\, F_k P_k F_k' + q_2 J_k P_k J_k' + q_2 K_{k+1} R K_{k+1}' + Q. \tag{4.15}$$

Then, from (4.13), V_{k+1} with (4.14) and (4.15) becomes

$$V_{k+1} = (J_k \tilde{x}_k + z_{k+1})' P_{k+1}^{-1} (J_k \tilde{x}_k + z_{k+1})$$
$$= \tilde{x}_k' J_k' P_{k+1}^{-1} J_k \tilde{x}_k + 2 z_{k+1}' P_{k+1}^{-1} J_k \tilde{x}_k + z_{k+1}' P_{k+1}^{-1} z_{k+1}. \tag{4.16}$$

By (4.15) and the matrix inversion formula, we know that

$$J_k' P_{k+1}^{-1} J_k = J_k' \{(1 - q_2) F_k P_k F_k' + q_2 J_k P_k J_k' + q_2 K_{k+1} R K_{k+1}' + Q\}^{-1} J_k$$
$$= [\, q_2 P_k + J_k^{-1} \{(1 - q_2) J_k F_k P_k F_k' + q_2 K_{k+1} R K_{k+1}' + Q\} (J_k')^{-1}]^{-1}$$
$$= P_k^{-1/2} [\, q_2^{-1} \mathrm{I} - \{\, q_2 \mathrm{I} + q_2 P_k^{1/2} J_k' \, [(1 - q_2) F_k P_k F_k'$$
$$+ q_2 K_{k+1} R K_{k+1}' + Q]^{-1} q_2 J_k P_k^{1/2} \}^{-1} \,] P_k^{-1/2}$$
$$\le (q_2 P_k)^{-1} \{ 1 - (1 + \| q_2 J_k' P_k J_k [(1 - q_2)$$
$$\cdot F_k P_k F_k' + q_2 K_{k+1} R K_{k+1}' + Q]^{-1} \|)^{-1} \}$$
$$\le (q_2 P_k)^{-1} \{ 1 - [1 + \| (q_2 F_k P_k F_k' + Q) Q^{-1} \|]^{-1} \}$$
$$\le (q_2 P_k)^{-1} - \frac{1}{2 + \| Q^{-1} \| \| q_2 F_k P_k F_k' \|} (q_2 P_k)^{-1}. \tag{4.17}$$

To derive the third equality in (4.17) the matrix inversion fo r-mula has been used. To derive the second inequality,

$$q_2 J_k' P_k J_k \le q_2 F_k P_k F_k' + Q$$

and

$$\{(1 - q_2) F_k P_k F_k' + q_2 K_{k+1} R K_{k+1}' + Q\}^{-1} \le Q^{-1}$$

have been used.

Substituting (4.17) into (4.16) we get

$$V_{k+1} \le \tilde{x}_k' \left[(q_2 P_k)^{-1} - \frac{1}{2 + \| q_2 Q^{-1} F_k P_k F_k' \|} (q_2 P_k)^{-1} \right] \tilde{x}_k$$
$$+ 2 z_{k+1}' P_{k+1}^{-1} J_k \tilde{x}_k + z_{k+1}' P_{k+1}^{-1} z_{k+1}$$
$$\le q_2^{-1} V_k - \frac{1}{2 + \| Q^{-1} \| \| q_2 F_k P_k F_k' \|} q_2^{-1} V_k$$
$$+ 2 z_{k+1}' P_{k+1}^{-1} J_k \tilde{x}_k + z_{k+1}' P_{k+1}^{-1} z_{k+1}. \tag{4.18}$$

Now, if using the elementary inequality $2|xy| \le x^2 + y^2$, the second term of (4.18) yields:

$$2 \,|\, z_{k+1}' P_{k+1}^{-1} J_k \tilde{x}_k \,| \le 2 \| z_{k+1}' P_{k+1}^{-1/2} \| \| P_{k+1}^{-1/2} J_k \tilde{x}_k \|$$
$$\le 2 z_{k+1}' P_{k+1}^{-1} z_{k+1} (\, 2 + \| Q^{-1} \| \| q_2 F_k P_k F_k' \|)$$
$$+ \frac{\tilde{x}_k' J_k' P_{k+1}^{-1} J_k \tilde{x}_k}{2(\, 2 + \| Q^{-1} \| \| q_2 F_k P_k F_k' \|).}$$
$$\le 2 z_{k+1}' P_{k+1}^{-1} z_{k+1} (\, 2 + \| Q^{-1} \| \| q_2 F_k P_k F_k' \|)$$
$$+ \frac{q_2^{-1} V_k}{2(\, 2 + \| Q^{-1} \| \| q_2 F_k P_k F_k' \|).} \tag{4.19}$$

In deriving the last term in (4.19), the following relation from (4.13) and (4.17) has been used:

$$\tilde{x}_k' J_k' P_{k+1}^{-1} J_k \tilde{x}_k \le q_2^{-1} V_k .$$

By using $z_{k+1} = -K_{k+1} n_{k+1} + w_k$ it follows that

$$\begin{aligned}
&(z_{k+1}' P_{k+1}^{-1} z_{k+1})(z_{k+1}' P_{k+1}^{-1} z_{k+1}) \\
&= z_{k+1}' P_{k+1}^{-1} z_{k+1} \| P_{k+1}^{-1/2} (-K_{k+1} n_{k+1} + w_k) \|^2 \\
&\le z_{k+1}' P_{k+1}^{-1} z_{k+1} [(K_{k+1}' P_{k+1}^{-1} K_{k+1}) \| n_{k+1} \|^2 + P_{k+1}^{-1} \| w_k \|^2] \\
&\le O([K_{k+1}' P_{k+1}^{-1} K_{k+1}] \| n_{k+1} \|^2) + O(\| w_k \|^2) \\
&\le O(\| n_{k+1} \|^2 + \| w_k \|^2).
\end{aligned} \tag{4.20}$$

Consequently, substituting (4.19)~(4.20) into (4.18) we get

$$\begin{aligned}
V_{k+1} &= q_2^{-1} V_k - \frac{1}{4 + 2\| Q^{-1} \| \| q_2 F_k P_k F_k' \|} q_2^{-1} V_k \\
&\quad + z_{k+1}' P_{k+1}^{-1} z_{k+1} \left(5 + \frac{2 \| q_2 F_k P_k F_k' \|}{\| Q \|} \right) \\
&\le q_2^{-1} V_k - \frac{1}{4 + 2 \| Q^{-1} \| tr(q_2 F_k P_k F_k')} q_2^{-1} V_k \\
&\quad + O\{ (\| w_k \|^2 + \| n_{k+1} \|^2) \| q_2 F_k P_k F_k' \| \}.
\end{aligned} \tag{4.21}$$

■

Remark 5: Similarly to the proof of (4.10) or (4.11) it can be shown that

$$E[S_{m+1} | \Im_{mh-1}] \le \left(1 - \frac{q_2 d \| Q \|}{2d(\| R \| + h \| Q \|)} \right) S_m + O(1) , \ \forall m \ge 0,$$

where $S_m = \sum_{k=(m-1)h}^{mh-1} tr(P_{k+1})$.

Remark 6: It is noted that $\{a_k\}$ in Lemma 4 is the form of $a_k = 4 + e\ tr(q_2 F_k P_k F_k)$, where $e = 2 \| Q^{-1} \|$. Therefore, if $\Omega(n,k)$ is defined as

$$\Omega(n+1,k) = \left(q_2^{-1} - \frac{q_2^{-1}}{4 + e tr(q_2 F_n P_n F_n')} \right) \Omega(n,k), \qquad \Omega(k,k) = 1, \quad \forall n \ge k \ge 0 \tag{4.22}$$

then

$$E[\Omega(n+1,k)] \le M g^{\, n-k+1}, \quad \forall n \ge k \ge 0, \quad 0 < g < 1, \quad M < \infty . \tag{4.23}$$

Finally, with these lemmas we are able to prove the main result of this paper.

Proof of Theorem 1: From (4.22) and Lemma 5 it follows that

$$V_n \le W(n,0) V_0 + O\left(\sum_{k=0}^{n-1} W(n,k) \| q_2 F_k P_k F_k' \| (\| n_{k+1} \|^2 + \| w_k \|^2) \right).$$

So by the Minkowski inequality we have

$$\begin{aligned}
\{ E(V_n)^{4/3} \}^{3/4} &\le \Big(E[\Omega(n,0) V_0 \\
&\quad + O\left(\sum_{k=0}^{n-1} \Omega(n,k) \| q_2 F_k P_k F_k' \| (\| n_{k+1} \|^2 + \| w_k \|^2) \right)]^{4/3} \Big\}^{3/4} \\
&\le O\Bigg(\sum_{k=0}^{n-1} [E\{ \Omega(n,k) \| q_2 F_k P_k F_k' \| \\
&\qquad \cdot (\| n_{k+1} \|^2 + \| w_k \|^2) \}^{4/3}]^{3/4} \Bigg) \\
&\quad + [E\{ \Omega(n,0) V_0 \}^{4/3}]^{3/4} .
\end{aligned} \tag{4.24}$$

Now, by the Holder inequality, Lemma 2, Assumption A2, Assumption A3, and the fact that $\Omega(n,k) \le 1$, we know that

$$\begin{aligned}
&E\{ W(n,k) \| q_2 F_k P_k F_k' \| (\| n_{k+1} \|^2 + \| w_k \|^2) \}^{4/3} \\
&\le 2^{1/3} E[\{ W(n,k) \}^{4/3} \| q_2 F_k P_k F_k' \|^{4/3} \\
&\qquad \cdot (\| n_{k+1} \|^{8/3} + \| w_k \|^{8/3})] \\
&\le O([E\{ W(n,k) \}^4]^{1/3} \{ E(\| q_2 F_k P_k F_k' \|^4) \}^{1/3} \\
&\qquad \cdot \{ E(\| n_{k+1} \|^r + \| w_k \|^r) \}^{8/3r}) \\
&\le O((q_2^{4/3}) [E\{ W(n,k) \}]^{4/3} \{ E(\| n_{k+1} \|^r + \| w_k \|^r) \}^{8/3r}) \\
&\le O((q_2^{4/3}) [s_r]^{8/3r} [E\{ W(n,k) \}]^{4/3}).
\end{aligned}$$

Hence, it follows from (4.23) and (4.24) that

$$\begin{aligned}
&\limsup_{n \to \infty} [E\{ V_n \}^{4/3}]^{3/4} \\
&\le \limsup_{n \to \infty} \Bigg\{ O\left(\sum_{k=0}^{n-1} [(q_2)^{4/3} [s_r]^{8/3r} [E\{ \Omega(n,k) \}]^{4/3}]^{3/4} \right) \\
&\qquad + \{ E(\Omega(n,k) V_0)^{4/3} \}^{3/4} \Bigg\} \\
&= O(q_2 [s_r]^{2/r}).
\end{aligned}$$

Therefore, we have the result as follows:

$$\begin{aligned}
\limsup_{n \to \infty} E[\| \tilde{x}_n \|^2] &\le \limsup_{n \to \infty} E(\| P_n^{1/2} \|^2 \| P_n^{-1/2} \tilde{x}_n \|^2) \\
&\le \limsup_{n \to \infty} E(\| P_n \| V_n) \\
&\le \limsup_{n \to \infty} (E \| P_n \|^4)^{1/4} \{ E(V_n)^{4/3} \}^{3/4} \\
&\le O(q_2 [s_r]^{2/r}).
\end{aligned}$$

Now, it remains to prove the second result of Theorem 1. By Lemma 5, it is evident that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{q_2^{-1} V_k}{4 + e \ tr(q_2 F_k P_k F_k')} \\
&= \sum_{k=0}^{n-1} (q_2^{-1} V_k - V_{k+1}) + O\left(\sum_{k=0}^{n-1} \| q_2 F_k P_k F_k' \| \{ \| n_{k+1} \|^2 + \| w_k \|^2 \} \right) \\
&= O(1) + O\left(\sum_{k=0}^{n-1} \| q_2 F_k P_k F_k' \| \{ \| n_{k+1} \|^2 + \| w_k \|^2 \} \right)
\end{aligned}$$

So by the Schwarz inequality, Assumption A2 and A3, and Lemma 3,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{q_2^{-1} V_k}{4 + e \ tr(q_2 F_k P_k F_k')}$$

$$\le O\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\|q_2F_kP_kF_k'\|\left\{\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}(\|n_{k+1}\|^2+\|w_k\|^2)\right\}\right)$$

$$\le O\left(q_2\cdot\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}(\|n_{k+1}\|^2+\|w_k\|^2)\right)$$

$$\le O(q_2(m_4)^{1/2}).$$

Consequently, by this and Lemma 3 it follows that ($f=1+e$)

$$\sum_{k=0}^{n-1}\|\tilde{x}_k\|=\sum_{k=0}^{n-1}\frac{\|\tilde{x}_k\|}{[4+f\{tr(q_2F_kP_kF_k')\}^2]^{1/2}}[4+f\{tr(q_2F_kP_kF_k')\}^2]^{1/2}$$

$$\le O\left(n^{1/2}\left\{\sum_{k=0}^{n-1}\frac{\|\tilde{x}_k\|^2}{tr(q_2F_kP_kF_k')\{4+e\,tr(q_2F_kP_kF_k')\}}\right\}^{1/2}\right)$$

$$\le O\left(n^{1/2}\left\{\sum_{k=0}^{n-1}\frac{q_2^{-1}V_k}{4+e\,tr(q_2F_kP_kF_k')}\right\}^{1/2}\right)$$

$$=O(n(q_2)^{1/2}(m_4)^{1/4}).$$

Therefore, we obtain the result in this paper as follows:

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\|\tilde{x}_k\|\le O((q_2)^{1/2}(m_4)^{1/4}).\qquad\blacksquare$$

V. Conclusions

The PDAF with a modified Riccati equation, which was developed for solving the uncertainty problem regarding the origin of a measurement, is known to show better tracking performance than the standard Kalman filter in a cluttered environment. In this paper, using the Lyapunov function approach, the stability of the PDAF with a modified Riccati equation was analytically proved. It was shown that if the measurement sequence belongs to a gate s-algebra, the information reduction factor is chosen adequately takes the value between 0 and 1, and the process and observation noises are bounded, then the state estimation error is exponentially bounded and the bounds are determined by the bounds of the process and observation noises.

It is expected that the analytic methods developed in this paper can be extended to the stability and performance analyses of the PDAF with the stochastic Riccati equation and the PDAF with interacting multiple models.

References

[1] Y. Bar-Shalom and E. Tse, "Tracking in a cluttered environment with probabilistic data association," *Automatica*, vol. 11, no. 5, pp. 451-460, 1975.

[2] Y. Bar-Shalom and T. E. Fortmann, *Tracking and Data Association*, Academic Press, New York, 1988.

[3] H. Bauer, *Probability Theory and Elements of Measure Theory*, Academic Press, New York, 1981.

[4] M. Boutayeb, H. Rafaralahy, and H. Darouach, "Convergence analysis of the extended Kalman filter used as an observer for nonlinear discrete-time system," *IEEE Trans. Automatic Control*, vol. 42, no. 4, pp. 581-586, 1997.

[5] J. Chen, H. Leung, T. Lo, J. Litva, and M. Blanchette, "A modified probabilistic data association filter in a real clutter environment," *IEEE Trans. Aerospace and Electronic Systems*, vol. 32, no. 1, pp. 300-313, 1996.

[6] C. K. Chui and G. Chen, *Kalman Filtering with Real-time Applications*, Springer-Verlag, New York, 1991.

[7] T. E. Fortmann, Y. Bar-Shalom, M. Scheffe, and S. B. Gelfand, "Detection thresholds for tracking in clutter: A connection between estimation and signal processing," *IEEE Trans. Automatic Control*, vol. AC-30, no. 3, pp. 221-228, 1985.

[8] A. Gelb, *Applied Optimal Estimation*, MIT Press, Messachusetts, 1974.

[9] G. C. Goodwin and K. S. Sin, *Adaptive Filtering Prediction and Control*, Prentice- Hall, Englewood Cliffs, NJ, 1982.

[10] L. Guo, "Estimating time-varying parameter by the Kalman filter based algorithm: Stability and convergence," *IEEE Trans. Automatic Control*, vol. 35, no. 2, pp. 141-147, 1990.

[11] L. Guo, "Stability of recursive stochastic tracking algorithms," *SIAM J. Control and Optimization*, vol. 32, no. 5, pp. 1195-1225, 1994.

[12] L. Guo and L. Ljung, "Performance analysis of general tracking algorithms," *IEEE Trans. Automatic Control*, vol. 40, no. 8, pp. 1388-1402, 1995.

[13] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1972.

[14] D. J. Kershaw and R. J. Evans, "A contribution to performance prediction for probabilistic data association tracking filters," *IEEE Trans. Aerospace and Electronic Systems*, vol. 32, no. 3, pp. 1143-1148, 1996.

[15] W. H. Kwon, K. S. Lee, and O. K. Kwon, "Optimal FIR filters for time-varying state-space models," *IEEE Trans. Aerospace and Electronic Systems*, vol. 26, no. 6, pp. 1011-1021, 1990.

[16] W. H. Kwon, Y. S. Suh, Y. I. Lee, and O. K. Kwon, "Equivalence of finite memory filters," *IEEE Trans. Aerospace and Electronic Systems*, vol. 30, no. 3, pp. 968-972, 1994.

[17] X. R. Li and Y. Bar-Shalom, "Stability evaluation and track life of the PDAF for tracking in clutter," *IEEE Trans. Automatic Control*, vol. AC-36, no. 5, pp. 588-602, 1991.

[18] L. Ljung, "Asymptotic behavior of the extended Kalman filter as a parameter for linear systems," *IEEE Trans. Automatic Control*, vol. AC-24, no. 1, pp. 36-50, 1979.

[19] K. Reif, S. Gunther, E. Yaz, and R. Unbehauen, "Stochastic stability of the discrete-time extended Kalman filter," *IEEE Trans. Automatic Control*, vol. 44, no. 4, 1999.

[20] V. Solo, "The stability of least mean square," *IEEE Trans. Signal Processing*, vol. 45, no. 12, pp. 3017-3026, 1997.

[21] V. Solo, "Stability of the Kalman filter with stochastic time-varying parameters," *IEEE Proceedings of the 35th Conference on Decision and Control*, Kobe, Japan, pp. 57-61, December, 1996.

[22] T. L. Song, "Observability of target tracking with bearings-only measurements," *IEEE Trans. Aerospace and Electronic Systems*, vol. 32, no. 4, pp. 1468-1471, 1996.

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